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Phase Space Analysis and Particle Structure in Field Theory

Jacques Magnen[1]

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Progress in the last few years in constructive field theory has included (1) the construction of three-dimensional, non-Abelian gauge theories in a finite volume, (2) the construction of models that are asymptotically free or have a nontrivial fixed point, (3) a new presentation of perturbation theory yielding old and new large-order estimates, and (4) asymptotic completeness for renormalizable models in the two-particle region; the existence of multiparticle structure equations in the Euclidean region, provided that the coupling constant goes to zero as the energy increases; these equations yield formally the asymptotic completeness. In obtaining most of these results the phase space expansion pioneered by Glimm and Jaffe plays a crucial role. The first part of this review is a description of the phase space expansion in the "Ecole Polytechnique" version; the second part is devoted to the construction of models, the third part treats perturbation theory; and the last part deals with asymptotic completeness.

KEY WORDS: Field theory; phase space analysis; renormalization group; (construction of) asymptotically free models; (construction of) non-Abelian gauge theories; (construction of) models with nontrivial fixed point; asymptotic completeness; particle structure.

1. THE PHASE SPACE EXPANSION

This is a high-temperature expansion on the coupling between phase space cells completed by a renormalization procedure; each phase cell corresponds roughly to one degree of freedom.

We describe the expansion for the φ^4 model in four dimensions chosen as a "toy" model. The Schwinger functions are given by $S = \underline{S}/Z$,

$$\underline{S}(x_1,..., x_n) = \int \varphi(x_1) \cdots \varphi(x_n) \exp\left[-\lambda \int \varphi(z)^4 \, dz \right] d\mu$$

[1] Centre de Physique théorique, Ecole Polytechnique F-91128, Palaiseau, France.

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where Z is the partition function and $d\mu$ is the Gaussian measure of covariance, or propagator

$$C(p) = 1/(p^2 + m^2)$$

In fact we consider the model with a momentum cutoff $\eta_\rho(p)$

$$\eta_\rho(p) = \exp[-(p^2 + m^2) M^{-2\rho}]$$

and a bare coupling constant λ_ρ.

The two main features of this theory are (1) the number of perturbation diagrams is $n!$ at order n because the n vertices of the perturbation can all correspond to the same degree of freedom; the expansion will thus be of finite order for each phase cell; and (2) the two- and four-point diagrams are ultraviolet (primitively) divergent; this is the reason for the renormalization, which concerns only two- and four-point subdiagrams.

1.1. Phase Space Cells

The phase space cells are defined by a momentum localization and by a space localization. The momentum localization is defined by slicing the propagator

$$C(p) = \sum_{i=1}^{\rho} C^i(p), \qquad C^i(p) = [\eta_i(p) - \eta_{i-1}(p)]\, C(p), \qquad \text{with} \quad \eta_0(p) \equiv 0$$

so that

$$0 \leqslant C^i(x-y) \leqslant 0(1)\, M^{2i} \exp(-M^i\, |x-y|)$$

Thus, the field φ is the sum of mutually orthogonal field φ^i of covariance C^i and of associated measure $d\mu_i$; the unnormalized Schwinger functions are rewritten as

$$\underline{S}(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_n} \int \varphi^{i_1}(x_1) \cdots \varphi^{i_n}(x_n)$$
$$\times \exp\left\{ -\lambda \int \left[\sum \varphi^i(z) \right]^4 dz \right\} \prod d\mu_i$$

where we see that the momentum slices are coupled by the exponential of the interaction.

For each i we make a partition $\mathbb{D}_i$ of the space in cubes of size M^{-4i}. The size of the squares is dual to the size of the momenta and we have

$$\sum_{\Delta \in \mathbb{D}_i} \exp[-M^i \operatorname{dist}(\Delta_0, \Delta)] \leqslant O(1), \qquad \text{independently of } i$$

$(i, \Delta \in \mathbb{D}_i)$ is a phase cell, and each field is localized in a phase cell.

The measure $d\mu$ couples phase cells having the same index (but different space localization) and the exponential of the interaction couples phase cells that are contained in one another but with different indices. Because of the above inequality all sums over cells in configuration space are done using the exponential decrease of the covariances. The bounds are then reduced to some kind of power counting: (1) M^i by a field of index i (C^i corresponds to the contraction of two fields); (2) M^{-4i} by a vertex summed in a cube of $\mathbb{D}_i$.

1.2. A High-Temperature Expansion between Phase Cells

(i) For each pair of cells of the same index i we make an expansion of their coupling by the propagator C^i. We generate this expansion by introducing one perturbation parameter for each pair of vertices:

$$C^i(x, y; s) = \sum_{\Delta,\Delta' \in \mathbb{D}_i} s_{\Delta,\Delta'}\, \Delta(x)\, \Delta'(y)\, Ci(x-y)$$

with $s_{\Delta,\Delta'} = s_{\Delta',\Delta}$ and for all Δ, $s_{\Delta,\Delta} = 1$, where $\Delta(x)$ is the characteristic function of Δ. (What we really do is more complicated due to some technicalities without interest here.) We have

$$C^i(x, y; s)\,|_{s \equiv 1} = C^i(x-y)$$

and at $s_{\Delta,\Delta'} = 0$ there is no coupling between Δ and Δ'. We generate the expansion by doing for each pair (Δ, Δ') of $\mathbb{D}_i$ a first-order Taylor expansion around $s_{\Delta,\Delta'} = 1$ of the measure $d\mu_i$,

$$\begin{aligned}
&\int Re^{-\varphi^4}\, d\mu(s)\,|_{s_{\Delta,\Delta'}=1} \\
&\quad = \int Re^{-\varphi^4}\, d\mu(s)\,|_{s_{\Delta,\Delta'}=0} \\
&\quad\quad + \int_0^1 ds \int dx\, dy\, C^i(x, y)\, \Delta(x)\, \Delta'(y) \frac{\delta}{\delta\varphi(x)} \frac{\delta}{\delta\varphi(y)} Re^{-\varphi^4}\, d\mu(s)
\end{aligned}$$

which can be graphically described by

$$^{\Delta}\square\ \ ^{\Delta'}\square = \underset{\substack{\text{with no propagator}\\ \text{between } \Delta \text{ and } \Delta'}}{^{\Delta}\square\ \ ^{\Delta'}\square} + \underset{\substack{\text{with at least one propagator}\\ \text{between } \Delta \text{ and } \Delta'}}{^{\Delta}\square\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\ ^{\Delta'}\square}$$

(ii) For each phase cell (i, Δ) we make an expansion on the coupling between phase cells with index smaller and with index bigger than i (and which have the same localization in configuration space). We generate this in the following way: for each $\Delta \in \mathbb{D}_i$ we introduce a perturbation parameter t_Δ and define the t dependence

$$[\varphi_h(x) + t_\Delta \varphi_l(x)]^4 + (1 - t_\Delta^4)\, \varphi_l(x)^4, \qquad x \in \Delta$$

where φ_h is the sum of the fields of index bigger than or equal to i; each φ^j, $j \geqslant i$, is by definition a *high-momentum field* (relative to Δ); and φ_l is the sum of the fields of index smaller than i and is by definition a *low-momentum field* (relative to Δ).

This t-dependent interaction interpolates between the coupled and the uncoupled cases: for $t_\Delta = 1$ it gives $[\varphi_h(x) + \varphi_l(x)]^4 = \varphi(x)^4$ and for $t_\Delta = 0$ it gives $\varphi_h(x)^4 + \varphi_l(x)^4$. We introduce this in the exponential of the interaction and make for each cell a Taylor expansion (to the fifth order) around $t = 1$. We obtain a sum of terms where there are in Δ (the cube of production) exactly 0, 1, 2, 3 or 4 vertices coupling φ_h and φ_l if $t_\Delta = 0$ and 5 or more if $t_\Delta \neq 0$. Each perturbation vertex has at least one and at most three low-momentum fields (resp. high-momentum fields).

A vertex that has fields in different cells connects them by definition. In this way we have explored all the structure of possible connections between cells.

If we can prove that each connection is small, i.e., that

$$\left| \sum_{\substack{\text{all connected sets} \\ \text{of } n \text{ cubes containing } \Delta_0}} (\text{contributions}) \right| \leqslant e^{-Kn}$$

with K large enough, then we have indeed constructed a convergent high-temperature expansion between phase cells. However, each connected subgraph cannot overlap with any other one; to relax this dependence, we make in each slice a Mayer expansion on the cubes, which is convergent if the above convergence condition holds.

An almost local subgraph is by definition connected and such that all its external fields are of indices strictly lower than the ones of its internal fields. *An almost local subgraph of index i*, by definition, contains at least one field of index i and all its (internal) fields are of index bigger than or equal to i.

2.3. The Bounds

The result of the expansion of $\underline{S}$ is a sum of terms composed of integrals over s and t parameters of the product of fields belonging to perturbation vertices, of integrals over the localization of these vertices, of the exponential of the interaction (which depends on t's), and of the integration over the fields with the measure $d\mu(s)$. If R is a product of fields, a bound over the field integration is obtained using

$$\left|\int Re^{-\varphi^4}\,d\mu\right| \leqslant \left|\int R^2\,d\mu\right|^{1/2} \sup_{\varphi} e^{-\varphi^4}$$

This bound is too crude for our purpose: it is possible that all the t perturbations relative to cubes of $\mathbb{D}_j$ produce vertices with three fields in the cubes of $\mathbb{D}_i$, $i \ll j$, containing them; the situation with one such vertex per cube of $\mathbb{D}_j$ corresponds to $3M^{4(j-i)}$ fields in the corresponding cube of $\mathbb{D}_i$ and the integration of these fields leads to a number of contraction schemes of order $[M^{6(j-i)}]^{\text{number of vertices}}$, and for each vertex the factor $M^{6(j-i)}$ cannot be compensated by the volume M^{-4j} of the localization cube of the vertex. However, if in R the number of fields per phase cell is uniformly bounded, then the number of contractions is controlled by the exponential decrease of the propagator in configuration space (it is sufficient to sum over the localization cube of the field that is contracted). An immediate consequence is that the integration over the high-momentum fields gives

$$\int \left(\prod \text{high-momentum fields}\right)^2 d\mu$$

$$\leqslant \prod_i \{O(1)\, M^i \text{ by high-momentum field of index } i\}$$

The propagators associated with the low-momentum fields have an exponential decrease, which is not related to the size of the production cube. To bound them, we use the positivity of the interaction. A low-momentum field in Δ is almost constant in Δ in fact:

$$\varphi_l(x) = \varphi_\Delta + \delta\varphi(x), \qquad \text{with} \quad \varphi_\Delta = \frac{1}{|\Delta|}\int_\Delta \varphi(x)\,dx, \quad \delta\varphi = \varphi - \varphi_\Delta$$

where $\delta\varphi(x)$ can be considered as a high-momentum field. Using the Hölder inequality, we obtain

$$|\varphi_\Delta| \leqslant \lambda^{-1/4}\,|\Delta|^{-1/4}\left[\lambda \int_\Delta \varphi(x)^4\,dx\right]^{1/4}$$

and for $\Delta \in \mathbb{D}_i$ $|\Delta|^{-1/4} = M^i$, so that using the bound

$$\left|\int \prod (\text{fields } \varphi_h, \delta\varphi, \varphi_\Delta)\, e^{-\varphi^4}\,d\mu\right|$$

$$\leqslant \left|\int \prod \{\varphi_h, \delta\varphi\}^2\,d\mu\right|^{1/2} \sup_{\varphi} \prod_{\Delta} \{\varphi_\Delta\}\, e^{-\varphi^4}$$

and the fact that there is a finite number of fields per cell, we obtain the bound over the field integration given by the product of (1) M^i by high-momentum field of index i, (2) M^i by low-momentum field produced in a cube of index i, and $\lambda^{1/4}$ by perturbation vertex (because there is at most three low-momentum fields per perturbation vertex, yielding at most $\lambda^{-3/4}$, which together with the λ of the vertex gives $\lambda^{1/4}$).

Remark. For fermions there is no problem with the low-momentum fields, since the Pauli principle prevents the accumulation of fields in the same phase cell. In practice we use the fact that the Gaussian integration over fermions generates determinants; in such a determinant each column is associated to a fermion and each line to an antifermion or vice versa; the Pauli principle is reflected in the fact that a determinant with two identical columns (or lines) is zero.

It remains to bound the integration over the vertex localization. We proceed inductively from the slice ρ to the slice 1. The idea is to sum over each vertex using the strongest possible exponential decrease. We thus begin with the slice ρ; for each connected component there is a tree of propagators (coming from the cluster expansion) connecting the cubes of the component, i.e., having at least one field of index ρ and localized in it. Keeping one vertex fixed, we sum the others relative to it; the exponential decrease allows us to sum on the cubes containing the vertices and the sum of each vertex in the cube of $\mathbb{D}_\rho$ that contains it gives a factor $M^{-4\rho}$. More generally, each vertex summed at step i yields a factor M^{-4i}. For one connected subgraph contained in momentum slice i and having v vertices and e external fields ($4v-e$ internal fields), we obtain the bound

$$[O(1)\,\lambda^{1/4}]^v\,(M^{-4i})^{v-1}\,(M^i)^{4v-e}=[O(1)\,\lambda^{1/4}]^v\,M^{(4-e)i}$$

The general result is

$$\prod_{\text{vertices}}[O(1)\,\lambda^{1/4}]\prod_i\prod_{G_i^l}M^{[4-e(i,l)]}$$

where the G_i^l are the maximal almost local subgraphs made of fields of index bigger than or equal to i and $e(i,l)$ is the number of external fields of G_i^l. In particular, the expansion is convergent if, for all subgraphs G_i^l, $e(i,l)$ is bigger than 4. This is obviously not always the case, but the renormalization procedure makes a rearrangement of terms such that in the new expansion all the $e(i,l)$ are bigger than 4. The perturbation relative to the coupling of different momentum slices was pushed to the fifth order to factorize the two- and four-point almost local subgraphs; this allows us to compute explicitly each two- and four-point almost local subgraph.

Let us summarize the main point up to now of the expansion.

The perturbation expansion has two parts: (i) The perturbation of the propagators, which is an expansion in configuration space and is local in momentum space; and (ii) the perturbation of the interaction, which is an expansion in momentum space and is local in configuration space.

The bounds also have two parts: (i) the integration on the high-momentum fields, which is a bound local in momentum space and gives estimates on the coupling between different regions of the configuration space; and (ii) the domination of the low-momentum fields, which is local in configuration space, but nonlocal in momentum space.

This "simple" factorization in local aspects but in different spaces allows the above "natural" construction of the expansion.

1.4. Renormalization

Let us look only at the four-point subgraphs. Let $F_\rho(x_1,..., x_4)$ be the sum over all the almost local subgraphs of index ρ having four external legs $\varphi(x_1),..., \varphi(x_4)$ (of index smaller than ρ). If there are n such subgraphs and if we sum independently on them we must compensate with an $1/n!$; in fact, we obtain an exponential of this term; we can then associate each F_ρ with a coupling constant λ_ρ and look at the sum

$$\int F_\rho(x_1,..., x_4)\, \varphi(x_1) \cdots \varphi(x_4)\, dx_1 \cdots dx_4 - \int \lambda_\rho \varphi(x)^4\, dx$$

$$= \int F_\rho(x_1,..., x_4)[\varphi(x_1) \cdots \varphi(x_4) - \varphi(x_1)^4]\, dx_1 \cdots dx_4$$

$$- \int (\lambda_\rho + \delta\lambda_\rho)\, \varphi(x)^4\, dx$$

with

$$\delta\lambda_\rho = -\int F_\rho(x_1, x_2, x_3, x_4)\, dx_2\, dx_3\, dx_4$$

which is independent of x_1 by translation invariance. We set $\lambda_{\rho-1} = \lambda_\rho + \delta\lambda_\rho$. The first term on the lhs above is regularized; to see this, we rewrite the integrand:

$$\begin{aligned} F_\rho(x_1,..., x_4)\{&\varphi(x_1)[\varphi(x_2) - \varphi(x_1)]\, \varphi(x_3)\, \varphi(x_4) \\ &+ \varphi(x_1)\, \varphi(x_2)[\varphi(x_3) - \varphi(x_1)]\, \varphi(x_4) \\ &+ \varphi(x_1)\, \varphi(x_2)\, \varphi(x_3)[\varphi(x_4) - \varphi(x_1)]\} \end{aligned}$$

Then we express the differences of fields:

$$\varphi(x_k)-\varphi(x_1)=(x_k-x_1)\,\nabla\varphi[x_k+\vartheta(x_1-x_k)], \qquad 0\leqslant\vartheta\leqslant 1$$

Because F_ρ is connected and of momentum M^ρ it has an exponential decrease in x_k-x_1 and we have

$$|x_k-x_1|\exp(-M^\rho\,|x_k-x_1|)\leqslant O(1)\,M^{-\rho}$$

On the other hand, a gradient applied on a field of index j gives a factor corresponding to the magnitude of the momentum: M^j. We conclude that with regard to the bounds the regularizing operation is a multiplication by $M^{-\rho}M^j$, $j<\rho$; we can describe it as the replacing of a field of index ρ by a field of lower index; the "regularized F_ρ" now has five external fields as far as the bounds are concerned.

By induction we do this for all the almost local subgraphs with four external fields, which are then regularized; the price to pay is that the coupling constant is now an effective one, which is equal to λ_i for the vertices whose biggest field index is i and

$$\lambda_i=\lambda_\rho+\delta\lambda_\rho+\cdots+\delta\lambda_{i+1}$$

and $\delta\lambda_j$ is the zero-momentum value of the sum of all the almost local subgraphs of index j (the renormalization procedure being already performed for the indices ρ to $j+1$).

A similar procedure is applied to the two-point functions.

After the renormalization we have that $e(i,l)>4$ for all G_i^l; however, the factors $O(1)\,\lambda^{1/4}$ by vertex are replaced for a vertex of biggest field index i by $O(1)\,\lambda_i^{1/4}$; the convergence of the expansion thus depends on the λ_i being uniformly small, λ_1 (which in our case is near the renormalized coupling constant λ_{ren}) being different from zero in order to have a nontrivial theory. Theories satisfying this condition are said to be asymptotically safe. In particular, if $\delta\lambda_i>0$ for all i, then $\lambda_i>\lambda_{i+1}$ and the theory is asymptotically safe. If $\lambda_i\to 0$ as $i\to\infty$, the theory is asymptotically free.

2. CONSTRUCTION OF MODELS

The renormalizable, asymptotically free models constructed in ways similar to the one described above are as follows:

1. The massless φ_4^4 model with an ultraviolet cutoff.[8,13] For a lattice cutoff this is a model of Wiener paths (on the lattice) in four dimensions with a repulsive interaction. The model is infrared divergent and the

momentum slices are $(M^{-i}, M^{-(i-1)})$ instead of $(M^{(i-1)}, M^{i})$; the bare mass m is zero and one must compute the mass counterterm, which makes the theory massless; it is formally the sum of the one-particle, two-point irreducible graphs. One can compute this in terms of "one-particle irreducible" Mayer graphs; this allows a simple proof of the Borel summability of the theory in the bare coupling constant[8]; this proves in particular that the theory is uniquely defined by its perturbation expansion.

2. The massive Gross–Neveu model, which is a multicomponent $(N \geqslant 2)$ fermionic theory with a quartic interaction, satisfies all the Wightman axioms.[9,14] It is somewhat simpler than φ_4^4 because there is no need for the domination of the low-momentum fields, the theory being purely fermionic. Its renormalized perturbation series is Borel-summable.[4]

3. The nonlinear σ-model in the hierarchical approximation (in which by definition there is no coupling between phase cells of the same index).[16]

All these models have in common that the first-order contribution to $\delta\lambda$ is positive. If all λ_i are small enough so that $\delta\lambda_i$ is mainly given by this first-order contribution, then $\delta\lambda_i > 0$ and as a consequence

$$\lambda_i < \lambda_{i-1} < \cdots < \lambda_1$$

The biggest λ is then λ_1 and it suffices to choose $\lambda_1 > 0$ small enough to obtain that all λ_i remain small.

Asymptotically safe, nonrenormalizable models (thus having a nontrivial fixed point) have been constructed:

1. The planar φ_4^4 theory with a negative coupling constant in $4+\varepsilon$ dimensions. The theory with cutoff is defined by its perturbation series and one does a phase space expansion on the perturbation series.[6]

2. The Gross–Neveu model in $2+\varepsilon$ dimensions.[15]

The Gross–Neveu model in three dimensions with a large number N of components, which also has a nontrivial fixed point, can be similarly treated.[2] It must be thought of as a theory with a fermion field ψ and an ultralocal bosonic field σ with an interaction vertex $(1/N)^{1/2}\, \bar{\psi}\psi\sigma$. The theory at N infinite is exactly solvable and is a free theory in ψ and σ. The corrections make an interaction that is small like $(1/N)^{1/2}$ and is well suited to dominate the low-momentum σ fields.

In these models the coupling constant has a dimension $\lambda_i \approx aM^{-\alpha i}$, where a and α depend on the model; thus,

$$\delta\lambda_i = \lambda_i - \lambda_{i+1} \approx aM^{-\alpha i}(1 - M^{-\alpha}) > 0$$

On the other hand, the leading contribution to $\delta\lambda$ is positive and is explicitly computable (these are the two remarkable features of these theories); this allows the computation of a and α; the corrections are small, like powers of ε or $1/N$, which here are the small parameters that make the expansion convergent.

One of the main achievements of recent years is the construction by Balaban (for review see Ref. 1) of a non-Abelian gauge theory in three dimensions and in a finite volume as the limit of lattice approximations as the lattice spacing goes to zero.

It remains to do this for a four-dimensional, non-Abelian gauge theory; this program is in progress.[1,5]

An alternative program is to use a Euclidean-invariant cutoff which breaks the gauge invariance, and to restore it in the infinite-cutoff limit by ad hoc counterterms.[11] The cutoff is realized by adding a quadratic term in the Lagrangian, i.e., for a momentum cutoff of index ρ: $M^{-8\rho}A(\nabla)^{10}A$. This cutoff term stabilizes the theory, and the cutoff theory is finite without gauge choice. One can then define momentum slices using the cutoff term as the inverse of the propagator:

$$(C_\rho)^{\sim}=\frac{1}{p^{10}M^{-8\rho}+1}$$

where we have put a mass term as infrared cutoff,

$$C_\rho=\sum C^i, \qquad (C^i)^{\sim}=\frac{1}{p^{10}M^{-8(i+1)}+1}-\frac{1}{p^{10}M^{-8i}+1}$$

Each momentum slice corresponds to a momentum localization; we can then speak of phase cells.

Note that in a given phase cell in the small-field region (resp. in the big-field region) we are led to consider gauge transformation with momentum of the same scale as the one of the cell.

The main problem in gauge theories is that there is no natural separation in the Lagrangian of a quadratic part (which gives the propagator) and of an interaction part; indeed, these parts are not gauge-invariant. For example, in the Landau gauge ($\partial_\mu A_\mu=0$) the quadratic part is such that the theory is renormalizable, but the Lagrangian is not necessary large when the field is large, because of the Gribov ambiguities. This makes (naively?) the domination in the Landau gauge impossible. On the other hand, in the axial gauge ($A_0=0$) there is no Gribov ambiguity and the Lagrangian is well suited for domination; the propagator is, however, such that the theory seems nonrenormalizable. Finally, one sees that only large fields have to be dominated; indeed (the power counting is as for φ_4^4), it is necessary only to dominate low-momentum fields (in Δ) larger than $\Delta^{-1/4}$.

Balaban's recipe would consist, in our view, in splitting each phase cell into a small- and a big-field cell; then in perturbing locally in configuration space the coupling between small-fields cells (of different indices) and also the coupling between small- and big-fields cells; big-fields cells with neighboring indices (and contained in one another) are small (see below) and thus can be considered by convention connected. One can then define a local axial gauge for each big-fields cell (because there is no Gribov ambiguity with the axial condition). A Landau gauge can be defined elsewhere because there is no Gribov ambiguity in the small-field region. Then one does an ordinary phase cell expansion for the couplings between the small-fields cells. The big-fields cells are small just because the Lagrangian is large (the fact that the axial gauge is local makes possible an explicit link between the magnitude of the field and the one of the Lagrangian). In the small-fields cells it is an ordinary phase expansion of an asymptotically free theory without domination (the field being small). It is important to remark that as a consequence of our convention about gauge transformations in a cell (see above) the small- and big-field regions are (almost) stable.

3. PERTURBATION THEORY AND LARGE-ORDER ESTIMATES

One can apply the phase space expansion to the perturbation series and obtain a new description of the renormalization procedure and of the large-order estimates (first proved by de Calan and Rivasseau).[10,12] The bound on the sum of the nth-order diagrams, which for φ_4^4 is in $\mathrm{cst}^n n!$, has two origins: one is the bound on the perturbation series *in* each phase cell; it can be called the instanton effect, and reflects the fact that the number of diagrams is of order $n!$; and the other comes from expressing the λ_i in terms of λ_{ren} ($\lambda_{\text{ren}} = \lambda_1 + \delta\lambda_1$),

$$\lambda_i = \lambda_{\text{ren}} - \delta\lambda_1 - \cdots - \delta\lambda_i$$

and

$$\delta\lambda_i = \mathrm{cst} \cdot \lambda_{\text{ren}}^2 + \text{higher order terms}$$

so that

$$\lambda_i = \lambda_{\text{ren}} - (\mathrm{cst} \cdot \lambda_{\text{ren}}^2 \ln M^i) + \text{higher order terms}$$

Each effective coupling constant is given by its perturbative expansion. An nth-order diagram contributing to $\delta\lambda$ can contain as many as $n-1$

logarithmic divergences, which are controlled by the power convergence of the diagram

$$M^{-i}(\ln M^i)^n \leqslant \mathrm{cst} \cdot n!$$

It gives a contribution of order $n!$ coming from a single diagram; this is the renormalon effect. It is believed that the perturbation series of φ_4^4 is divergent and that the divergence is the one of the renormalon, but this remains to be proved. However, in the theory with a large number N of components one can prove this divergence and the existence of the first renormalon singularity[3] (following ideas of Parisi) because in this case the leading renormalon contribution (in the $1/N$ expansion) dominates the subleading contributions, which become smaller and smaller as N increases.

The instanton contribution has been explored using a "Lipatov analysis" by Brézin, Le Guillou, and Zinn-Justin. It gives the large-order behavior of the perturbation series for superrenormalizable models. For renormalizable models it should give the behavior of the sum of the perturbation series obtained by replacing each effective coupling constant by λ_{ren}. The leading Lipatov behavior has been confirmed for φ_3^4, which is superrenormalizable[20]; moreover, the existence of a singularity for the Borel transform of the perturbation series has been proved.[7]

For φ_4^4 only the upper bound was proved; let us give some indications.[21] For a theory with space and momentum cutoff it is straightforward to deduce Lipatov's upper bound from the Sobolev inequality, which reads

$$\int \varphi(x)^4 \, dx \leqslant K \left\{ \int [\nabla\varphi(x)]^2 \, dx \right\}^2$$

This inequality replaces one vertex by two vertices each time we use it. We thus get a factor corresponding to the sum over the phase cell which contains the extra vertex. If one does first (on the perturbation series) a phase space expansion (using this time zero Dirichlet boundary conditions to express the decoupling in the s-dependent propagator) and then uses the Sobolev inequality (on the vertices not derived by the expansion) in each connected set of phase cells, then the above sum is restricted to the connected set containing the φ^4 vertex. Either the set is very large and the bound is obvious and does not require the use of the Sobolev inequality, or it is small and the sum on the phase cells is under control.

Remark. The contributions coming from the vertices derived by the expansion are bounded, as in the phase space expansion, and because there are no two-, four-point subgraphs, there is no renormalization, hence no renormalon effect. Note, however, that because we are in perturbation theory, we cannot dominate the low-momentum fields of the derived vertices, so we integrate them with the Gaussian measure. Doing this in the phase space expansion for φ_4^4 leads (see above) to instanton divergences. Here, however, for the perturbation term of the nth order we have only to show that the part of the bound corresponding to the vertices derived by the phase expansion is small compared to $n!$.

4. ASYMPTOTIC COMPLETENESS AND MULTIPARTICLE STRUCTURE

Asymptotic completeness (AS) says that the Hilbert space of the states and of the asymptotic states are the same. In field theory this was proved for φ^4 in two dimensions and in the two- and three-particle regions. The proofs rely on the use of the Bethe–Salpeter equation (or its generalization to the three-particle case)

$$F = G + G \circ F \qquad \text{(BS)}$$

where F is the amputated connected four-point function, G is the Bethe–Salpeter kernel, which is the one-particle, irreducible, four-point function, and $\circ$ is the convolution with two effective propagators.

One then deduces from (BS) a discontinuity formula

$$F_+ - F_- = F_+ * F_-$$

where F_+, F_- are the boundary values of F from above and below the cut [which starts at $(2m_{\text{ph}})^2$] in the physical sheet and $*$ is the convolution on the mass shell. This discontinuity formula is equivalent to (AS). We briefly describe below some recent progress.

(i) Generalization of the previous results to asymptotically safe models.[19]

This is done using a modified (BS) equation introduced by Bros,

$$F = G_M + G_M \circ_M F$$

where $\circ_M$ is the convolution with two two-point functions with an ultraviolet cutoff M. The term G_M has the same decreasing properties as G, but dot not have a simple perturbative description. For the energies below M (the first momentum slice) we do the particle analysis done for φ_2^4, and above M (the momentum slices of index bigger than one) we just control the theory using the phase space expansion. Indeed, the propagators and phase cells corresponding to energies greater than M can be considered as one-particle irreducible, so that no particle analysis is needed.

(ii) The multiparticle structure in general energy regions. To generalize the (BS) equation to general energy regions is an unsolved problem. However, the "perturbative" (BS) equation

$$F = G + G \circ G + G \circ G \circ G + \cdots$$

was generalized by Iagolnitzer, and corresponding discontinuity formulas were conjectured[17]; they are believed to yield, in particular, asymptotic completeness. To establish these (perturbative) structure equations in the n-particle region, we use a cluster expansion where the Taylor expansion relative to each parameter s of the first momentum slice is pushed to order $n+1$. One can then read by inspection the equations for the unnormalized theory and for nonoverlapping kernels. A kind of Mayer expansion gives, then, the structure equations. The final expansion is a high-temperature expansion [between the cells of the slice 1, with now $M > (n+1)\,m$] where the n-particle structure in a given channel is extracted explicitly. The proof holds for the Euclidean theories and the coupling constant has to be smaller and smaller as the number of particles increases.[18]

To achieve (with a smaller and smaller coupling constant) the proof of asymptotic completeness, it remains to prove the discontinuity formulas. This is worked on, in three dimensions (for simplicity), for the weakly coupled φ^4 model, which is a theory without bound states.[4]

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